

New types of solvability in PT symmetric quantum theory

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ABSTRACT. The characteristic anti-linear (parity/time reversal, PT) symmetry of non-Hermitian Hamiltonians with real energies is presented as a source of two new forms of solvability of Schrödinger's bound-state problems. In detail we describe (1) their very specific semi-exact solvability (SES) and (2) their innovated variational tractability. SES technicalities are discussed via charged oscillator example. In a broader context, speculations are added concerning possible relationship between PT symmetry, solvability and superintegrability.

PT symmetric quantum mechanics: A brief introduction

Prehistory: Isospectral operators. Origins of the popular Bender's and Boettcher's PT symmetric quantum mechanics [B0, B2] lie in perturbation theory [CG, FG]. For an elementary illustration one may recollect the pioneering paper by Buslaev and Grecchi [BG] who proved the isospectrality of the Hermitian, spherically symmetric D -dimensional perturbed harmonic oscillator

$$H^{(PHO)}(g) = \frac{1}{2} \left(-\Delta + \sum_{j=1}^D x_j^2 \right) + g^2 \left(\sum_{j=1}^D x_j^2 \right)^2$$

(in its m -th partial-wave projection at any $m = 0, 1, \dots$) with its *non-Hermitian* "unstable" anharmonic partner(s) in one dimension,

$$H^{(UAO)}(g) = -\frac{d^2}{dz^2} + z^2(igz - 1)^2 - (D + 2m - 2)(igz - 1/2), \quad g > 0.$$

This means that the physics (and, in particular, the reality of energies) remains unchanged while the mathematics itself is significantly simplified when one weakens the Hermiticity of $H^{(PHO)}(g) = [H^{(PHO)}(g)]^\dagger$ to the mere PT symmetry of its spectrally equivalent partners [DT]. The latter operators commute with the product PT of parity and time reversal. In the modern language of review [M], one should rather speak about the pseudo-Hermiticity defined by the relation

$$H^{(UAO)}(g) = P \left[H^{(UAO)}(g) \right]^\dagger P.$$

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Recent progress: Exactly solvable examples. Paper [BG] extends the proof to the isospectrality between $H^{(PHO)}(g)$ and $H^{(UAO)}(g)$ to the imaginary couplings $g = ih$. The spectrum of the resulting self-adjoint double-well $H^{(UAO)}(ih)$ in one dimension (famous for its perceivably hindered perturbative tractability) may be then deduced from the “unstable” modification $H^{(PHO)}(ih)$ of the central quartic oscillator in D dimensions. The latter, non-Hermitian PT symmetric model is sufficiently simple in its ordinary differential “radial” representation of ref. [BG],

$$H_\varepsilon^{(URO)}(h) = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + r^2 - h^2 r^4,$$

$$r = r(x) = x - i\varepsilon, \quad x \in (-\infty, \infty), \quad h > 0, \quad \varepsilon > 0.$$

In the context of the present proceedings this operator admits an immediate re-interpretation as a specific perturbed and regularized form of the most common superintegrable oscillator $H_0^{(URO)}(0)$ of Smorodinnsky and Winternitz (SW, [FM, E]) in any number of dimensions and after its separation in cartesian coordinates.

The latter observation is inspiring since it is known that the concepts of (maximal) superintegrability and (exact) solvability are closely related at $\varepsilon = 0$ [T0]. In such a setting the weakening $\varepsilon > 0$ of the Hermiticity extends the class of the solvable models [Z0, Z1] as well as their interpretation in terms of Lie algebras [LC] and evokes a number of new open questions. We have addressed the first few of them in ref. [Z2] where we derived the exact spectrum and wave functions for the complexified SW oscillator $H_\varepsilon^{(URO)}(0)$. Our next move in this direction was devoted to the closely related complexified (though still separable) Calogero model of three particles [ZM] where an overlap of the integrability with PT symmetry acted as a generator of the new solvable self-adjoint Hamiltonians [ZT]. This experience makes the further study of PT symmetric models of (super)integrable type very promising. In the present paper, we are going to describe some new results achieved in this direction.

Running project: Partially solvable models. In place of the (not so easily feasible) analysis of the traditional quartic perturbations, we shall pay attention to the slightly simpler charged oscillators characterized, first of all, by their incompletely or quasi-exactly solvable¹ (QES) status. From the historical perspective, the acceptance of this concept was comparatively dramatic. In a prelude, various potentials have been shown partially solvable in terms of elementary functions. Thirty years ago, this period has been initiated by a two-page remark by André Hautot [H] who noticed that the charged oscillator possesses arbitrary finite multiplets of exact Sturmian solutions in two ($D = 2$) and three ($D = 3$) dimensions. The Hautot’s choice of the elementary wave-function ansatz

$$\varphi(r) = e^{-r^2/2 - g r} \sum_{n=0}^N h_n r^{n+\kappa}$$

(with arbitrary N) preceded the discoveries of the QES sextic oscillator [SB], of non-polynomial QES anharmonicities [WF] etc. During the “golden age” of the development of the subject, the existence of the common Lie-algebraic background of all the QES systems has been emphasized [T1]. A summary of the “state of the art” up to the early nineties has been offered by Alex Ushveridze in his monograph

¹term coined by A. Turbiner, meaning an elementary solvability for finite multiplets of states

[U] where the quartic polynomial oscillator is mentioned as a typical system *without* quasi-exact solvability.

A change of the approach to QES models has been initiated by Bender and Boettcher [B1] who revealed and described the QES solutions for the “unstable” quartic polynomial oscillators. Their construction has been extended to all the partial waves in ref. [Z3]. In paper [Z4] we made the next move and turned attention to PT symmetrization of the Hautot’s charged oscillator (cf. Appendix A). In the present continuation of this effort, we shall complete the picture by its extension, i.a., to all the partial waves (cf. Appendix B). In section 1 we describe our main result, viz., the quasi-even solutions which remained unnoticed in [Z4]. Unexpectedly, these “facilitated QES” states are available in *infinite* multiplets², all the elements of which are, basically, non-numerical. In a way emphasized in section 2, these sets may easily serve as certain non-standard bases in some “sufficiently large” subspaces of Hilbert space. For this reason, we suggest to call them semi-exactly solvable (SES, cf. Table 1).

TABLE 1. Tentative classification of solvability.

class	quasi – exact	semi – exact	exact
solutions available	finite set	infinitely many	all
range of couplings	restricted	restricted	any
illustrative example	$H^{(BBO)}(a, b)$	see below	$H_\varepsilon^{(URO)}(0)$

1. QUASI-EXACT SOLVABILITY ON COMPLEX CONTOURS

1.1. Three-term recurrences. Let us consider the Schrödinger equation

$$(1.1) \quad \left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{r^2(x)} + i \frac{F}{r(x)} + 2i b r(x) + r^2(x) \right] \psi_n(x) = E_n \psi_n(x)$$

$$\ell = \ell(L) = (L-1)/2, \quad L = D-2, D, D+2, \dots$$

It contrast to the Hautot’s QES problem of Appendix A it works with the purely imaginary charge and with a constant complex shift of coordinates $r = r(x) = x - i\varepsilon$ (cf. [A]). It also generalizes the $\ell = 0$ problem of ref. [Z4] so that we must employ the more powerful ansatz

$$(1.2) \quad \psi_n(r) = e^{-r^2/2 - i b r} \sum_{n=0}^N (i r)^{n-\ell} p_n.$$

²such a feature is much more characteristic for the completely solvable models

After its insertion, the differential form of our PT symmetric bound state problem (1.1) is replaced by the finite-dimensional matrix equation,

$$(1.3) \quad \begin{pmatrix} B_0 & C_0 & & & \\ A_1 & B_1 & C_1 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{N-1} & B_{N-1} & C_{N-1} \\ & & & A_N & B_N \\ & & & & A_{N+1} \end{pmatrix} \begin{pmatrix} p_0 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} = 0, \quad N < \infty$$

with the *real* matrix elements,

$$\begin{aligned} A_n &= b^2 + 2n - L - E, & B_n &= -(2n + 1 - L)b - F, \\ C_n &= (n + 1)(n + 1 - L), & L &= 2\ell + 1, \quad n = 0, 1, \dots \end{aligned}$$

The last line of (1.3) represents a separate condition $A_{N+1} = 0$ which gives our energies in closed form,

$$(1.4) \quad E = E_N = 2N + 2 - L + b^2, \quad N = 0, 1, \dots$$

The key consequence of this easy but important simplification lies in the emergence of a zero $C_{L-1} = 0$ in the upper diagonal of the square matrix of the simplified system (1.3). In Appendix B the presence of this zero enables us to construct the quasi-odd solutions at all ℓ .

1.2. The new, quasi-even solutions. All the coefficients in the polynomial wave functions (1.2) may be defined by closed formula (4.2) displayed in Appendix B. Let us now turn our attention to the quasi-even states. The superscript $(+)$ will be introduced to mark their even quasi-parity which may be characterized, for our present purposes, by the simple negation of the odd quasi-parity condition (4.3),

$$(1.5) \quad |p_0^{(+)}| + |p_1^{(+)}| + \dots + |p_{L-1}^{(+)}| > 0.$$

For quasi-even states, the vanishing matrix element $C_{L-1}^{(+)} = 0$ plays a different role. It separates again the two subsets of equations but the upper one ceases to be trivial. This means that the related subdeterminant must vanish,

$$(1.6) \quad S^{(+)} = \det \begin{pmatrix} B_0 & C_0 & & & \\ A_1 & B_1 & C_1 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{L-2} & B_{L-2} & C_{L-2} \\ & & & A_{L-1} & B_{L-1} \end{pmatrix} = 0.$$

Up to the exceptional, degenerate cases where eqs. (1.6) and (4.4) hold at the same time, we may drop the non-vanishing factor $S^{(+)} \neq 0$ and determine all the charges $F = F_{N,k}^{(+)}$ as roots of polynomial (1.6).

The $(+)$ -superscripted wave function coefficients are determined by eq. (4.2), this time in the full range of indices $j = N, N-1, \dots, 1$. As long as the dimension of these determinants grows with the difference $N-j$, we shall recommend a return

to the recurrences

$$\begin{pmatrix} -2N & \beta_1 - F & 4 - 2L & & & \\ & 2 - 2N & \beta_2 - F & 9 - 3L & & \\ & & \ddots & \ddots & \ddots & \\ & & & -4 & \beta_{N-1} - F & N^2 - NL \\ & & & & -2 & \beta_N - F \end{pmatrix} \begin{pmatrix} p_0^{(+)} \\ p_1^{(+)} \\ \vdots \\ p_{N-1}^{(+)} \\ p_N^{(+)} \end{pmatrix} = 0$$

to be read from below upwards³. The process must be initiated at $p_N^{(+)} \neq 0$. The vanishing $C_{L-1} = 0$ does not play any role in it.

Roots of the SES secular determinants

1.3. Dimension-independence. In every partial wave, the size L of the matrix in eq. (1.6) coincides with the degree of the resulting secular polynomial. This value does not change with the growth of the dimension $N + 1$ of the vector \vec{p} . Thus, the degree N of our polynomial wave functions enters our tridiagonal secular equation as a mere parameter. Containing extremely elementary matrix elements, the fully explicit form of this equation reads

$$\det \begin{pmatrix} (L-1)b - F & 1 - L & & & \\ -2N & (L-3)b - F & \ddots & & \\ & -2N + 2 & \ddots & 4 - 2L & \\ & & \ddots & (3-L)b - F & 1 - L \\ & & & -2(N+2-L) & (1-L)b - F \end{pmatrix} = 0$$

and specifies the family of the admissible charges $F = F_{N,k}^{(+)}$ at any integer $L > 0$ and index $k = 1, 2, \dots, L$. The first few partial waves are exceptional since their eigencharges may in principle be defined by closed formulae at $L = 1$, $L = 2$, $L = 3$ and $L = 4$. At the simplest choice of $L = 1$ (which may mean both the s -wave in three dimensions and an even state at $D = 0$), one does not obtain anything new. Secular equation (1.6) provides the single root $F_{N,1}^{(+)}$ which is equal to zero at all N . In the limit $\varepsilon \rightarrow 0$, the quasi-even $L = 1$ solutions converge to the well known Hermite polynomials. One just re-discovers the exactly solvable harmonic oscillator basis at even parity. We must re-emphasize that at any $L \geq 0$ the number of our quasi-even roots *does not change* with the growth of the dimension N . This is of paramount importance since the practical determination of eigencharges is performed for all the indices N at once. Each of these families numbered by k contains *infinitely many* elements numbered by the first subscript. This might facilitate their future applications (cf. section 2 below).

1.4. The first nontrivial generalization of oscillator basis: $L = 2$. Let us move to the index $L = 2$ giving the p -wave in two dimensions or to the s -wave in four dimensions. The related $\ell = 1/2$ wave functions may be chosen as compatible with the even-quasi-parity criterion (1.5),

$$p_0^{(+)}(N, k) = -\frac{1}{2N} [b + F_{N,k}] p_1^{(+)}(N, k), \quad k = 1, 2.$$

³we abbreviated $\beta_n \equiv -(2n + 1 - L)b$ for the sake of brevity

Secular equation (1.6) reads

$$\det \begin{pmatrix} b - F & -1 \\ -2N & -b - F \end{pmatrix} = F^2 - b^2 - 2N = 0$$

and has two roots

$$(1.7) \quad F_{N,1}^{(+)} = \sqrt{(b^2 + 2N)}, \quad F_{N,2}^{(+)} = -\sqrt{(b^2 + 2N)}.$$

[Note the contrast with the necessity of searching for roots of the polynomial (4.4) of the $N - L + 1$ st degree!] We encounter the non-vanishing charges for the first time. They grow with the increasing size of the shift b and with the quantum number N (i.e., with the energy). One finds their set more similar to the “ $F = 0$ line” of the harmonic oscillator than to the QES quasi-odd roots of Appendix B. Indeed, in the latter case the roots $F_{N,k}^{(-)}$ must be determined by the method which depends on N . In this sense we may now speak about the most natural and unique non-Hermitean generalization of the harmonic oscillator even-parity basis of $L_2(0, \infty)$.

1.5. Cardano charges at $L = 3$. At $L = 3$ the comparatively compact form of our secular equation

$$(1.8) \quad \det \begin{pmatrix} 2b - F & -2 & 0 \\ -2N & -F & -2 \\ 0 & -2N + 2 & -2b - F \end{pmatrix} = 0$$

enables us to search for the triplet of charges $\{F_{N,1}, F_{N,2}, F_{N,3}\}$ via the closed (so called Cardano) formulae. One of alternative strategies may consist in a transition from the SES eigencharges $F = F_{N,k}(b)$ to the inverse functions $b = b_{N',k'}(F)$. Such a trick lowers (by one) the degree of the secular polynomial at the odd values of L and, hence, extends the solvable class up to $L = 5$. A serious shortcoming of such an approach may be seen in the parallel deformation of the energies (1.4) which would change with the shift b .

In the similar spirit, another simplification of formulae may be based on the formal elimination of N . The consequences may be illustrated by the secular $L = 3$ determinant (1.8) which represents the mere linear problem for N , with the unique solution

$$(1.9) \quad N = -\frac{1}{8F} (4Fb^2 + 8b - F^3 - 4F) = \frac{1}{2} \left[\left(\frac{1}{F} + \frac{F}{2} \right)^2 - \left(b + \frac{1}{F} \right)^2 \right].$$

This formula suggests a simultaneous tuning of both the shifts $b_{[1,2]}$ and the related charges $F_{[1,2]}$. This may be achieved, say, by their hyperbolic re-parametrization with $F(t) = \sqrt{2}e^t$ and $\cosh t = \sqrt{N} \cosh \alpha(t)$ etc. In terms of the parameters $t = t_{N,k}$ and/or $\alpha = \alpha_{N,k}$, the spectrum of energies becomes deformed by the induced parametric dependence of $b = b_{[1,2]}(t) = \pm \sqrt{2 \cosh^2 t - 2N} - e^{-t}/\sqrt{2}$. Purely formally, the elimination of N might extend the use of closed formulae up to $L = 9$.

Higher partial waves

1.6. Numerical methods. A shortcoming of the present SES construction lies in the growth of its complexity at the large L . At $L \geq 5$ one already cannot generate closed formulae for eigencharges, and the purely numerical search for the

roots F is necessary at all the very large $L \gg 1$. In an illustration using $L = 3$, and $b = 5$ one gets the three eigencharges

$$F = \{10.757, -10.400, -0.35755\}$$

at the smallest possible $N = 2$. They smoothly grow to the values

$$(1.10) \quad F = \{89.98, -89.975, -0.0049407\}$$

evaluated at very large $N = 1000$. This type of calculation is very quick and its results exhibit a smooth N -dependence sampled in Table 2.

TABLE 2. N -dependence of eigencharges at $L = 4$ and $b = 5$

N	$F_{N,k}$			
3	-15.611	-5.927 9	4.888 7	16.651
30	-27.149	-9.290 9	8.929 4	27.511
300	-74.856	-24.984	24.936	74.904
3000	-232.82	-77.610	77.605	232.83
30000	-734.99	-245.00	245.00	734.99

1.7. Large N expansions. Any information concerning the eigenvalues $F = F(b)$ and/or $b = b(F)$ may shorten the necessary computations. At the intermediate $L = 5$ (which corresponds to the d -wave in three dimensions) we may eliminate, for example, the product $512 FN^\pm$ which is equal to

$$\left[-768b - 256Fb^2 + 768F + 40F^3 \pm 24\sqrt{(1024b^2 + 192bF^3 + 512F^2 + F^6)} \right]$$

and use this formula as a constraint. Fortunately, closed formulae of this type may also open the door to the perturbative methods.

Empirically, many numerically computed roots $F_{N,k}$ are very smooth functions of N , especially in the asymptotic domain where they may be well approximated by their available large- N estimates. For example, the exact result (1.10) already lies very close to its leading-order analytic estimate

$$F \approx \{\sqrt{8N}, -\sqrt{8N}, -b/N\} \approx \{89.44, -89.44, -0.005\}.$$

At the intermediate values of the dimension N , precision may be still insufficient but one can easily evaluate the large- N corrections. For illustration, let us return to eq. (1.9) and re-write it, in the case of its smallest root $F = -\hat{b}\hat{F}/N$, as the strictly equivalent formula

$$\hat{F} = 1 + \frac{\beta}{N^3} \hat{F}^3, \quad \beta = \frac{\hat{b}^3}{8b}, \quad \hat{b} = \frac{b}{1 + (b^2 - 1)/2N}, \quad L = 3.$$

It is suitable for iterations which represent our root as the following power series,

$$\hat{F} = 1 + \frac{\beta}{N^3} + 3 \frac{\beta^2}{N^6} + 12 \frac{\beta^3}{N^9} + 28 \frac{\beta^4}{N^{12}} + \dots$$

Also the other two $L = 3$ roots may be represented by the similar asymptotic series.

At the higher partial waves the same method works with a comparable efficiency. At $L = 4$, for example, the use of the variable $M = \sqrt{2N + b^2 - 2} > 0$ compactifies secular equation (1.6),

$$(F^2 - M^2) (F^2 - 9 M^2) = 36 - 48 b F .$$

In accord with Table 2, its asymptotically dominant $O(M^2)$ part determines the four distinct leading-order asymptotics of $F \sim \varrho M$ where $\varrho = \pm 1, \pm 3$. Once we re-normalize $F = \varrho M \sqrt{1 + R}$, the four exact charges obey the relation

$$R = \frac{48b\sqrt{1+R}}{8|\varrho| - 16 + \varrho^2 R} \frac{1}{\varrho M^3} + \frac{36}{8|\varrho| - 16 + \varrho^2 R} \frac{1}{\varrho^2 M^4} , \quad L = 4 .$$

Its iterations evaluate the correction term R with an astounding efficiency.

2. QUASI-VARIATIONAL SOLVABILITY ON MODEL SPACES

“Sufficiently large” finite subspaces in Hilbert space

2.1. Left and right SES solutions. A full-fledged applicability of the standard sets of QES wave functions is weakened by their incompleteness. Moreover, all the Hermitian QES Hamiltonians are usually interpreted as possessing just a finite multiplet of elementary bound states (at a given, special QES coupling) or Sturmians (cf. their example in Appendix B). For this reason, even the study of their own small perturbations is not easy at all [Z5]. Still, the sets of QES solutions themselves become large enough when we take into consideration *both* their fixed-coupling and fixed-energy subsets. In practice, such a philosophy proved useful for approximation purposes [BC].

In this section, we are going to return to the latter idea in the present broader, PT symmetric SES context. For the sake of definiteness let us re-consider Hamiltonian of eq. (1.1) as if it were defined at *any* charge F standing at the Coulomb-like force $W(r) = i r^{-1}$. Then we may denote $H = H(F) = H(0) + F W$ and, in the spirit of the Dirac’s bra and ket notation, abbreviate $|\psi_{M,k}^{(-)}\rangle = |M, k\rangle$. This enables us to re-write the special SES version of eq. (1.1) in shorthand,

$$(2.1) \quad H(0) |M, k\rangle + W |M, k\rangle F_{M,k} = |M, k\rangle E_M \\ k = 1, \dots, L, \quad M = 0, 1, \dots .$$

Our PT symmetric Hamiltonian is non-Hermitian so that its left and right eigenstates will differ in general. One has to complement eq. (2.1) by its counterpart

$$(2.2) \quad \langle\langle N, j | H(0) + F_{N,j} \langle\langle N, j | W = E_N \langle\langle N, j | , \\ j = 1, \dots, L, \quad N = 0, 1, \dots$$

where the same operators act to the left. After a return to the differential form of this problem (1.1), a transition between Hermitian-conjugate equations (2.1) and (2.2) may be re-interpreted as a certain reflection R in the space of the parameters,

$$(2.3) \quad R : \{F \longleftrightarrow -F, \quad b \longleftrightarrow -b, \quad \varepsilon \longleftrightarrow -\varepsilon\} .$$

In an illustration using $L = 2$, our SES states of section 1 may be represented by a discrete set of points in the energy-charge plane. The values of their coordinates

$(E_N, F_{N,k})$ are specified by the respective *closed formulae* (1.4) and (1.7). These points may be perceived as located on the two (viz. left and right) branches of a single hyperbolic curve. These two infinitely large multiplets are mutually related by the transformation R of eq. (2.3) so that $R|N, 1\rangle = |N, 2\rangle$ and *vice versa*.

2.2. Generalized Sturmian SES states as a basis. There is no *a priori* reason for a bi-orthogonality between our multi-indexed bras $\langle\langle N, k| \equiv \langle\langle A|$ and ket vectors $|N', k'\rangle \equiv |b\rangle$ so that their overlaps

$$Q_{A,b} = \langle\langle A|b\rangle$$

form a non-diagonal and asymmetric matrix in general. We have to assume that after a finite-dimensional truncation, this overlap matrix is invertible. Only in such a case we may define the inverse $R = Q^{-1}$ and introduce an identity projector in a “sufficiently large” subspace of Hilbert space,

$$(2.4) \quad I = \sum_{a \in J_{ket}, B \in J_{bra}} |a\rangle R_{a,B} \langle\langle B| .$$

In a constructive mood, let us return to equations (2.1) and (2.2) and imagine that they share all their eigen-energies and eigen-charges. This enables us to write down the following two alternative matrix equations

$$(2.5) \quad \langle\langle N, j| H(0) |M, k\rangle = \langle\langle N, j|M, k\rangle E_M - \langle\langle N, j| W |M, k\rangle F_{M,k}$$

$$(2.6) \quad \langle\langle N, j| H(0) |M, k\rangle = E_N \langle\langle N, j|M, k\rangle - F_{N,j} \langle\langle N, j| W |M, k\rangle$$

with $(N, j) \in J_{bra}$ and $(M, k) \in J_{ket}$. Their subtraction gives the constraint

$$(2.7) \quad (F_{M,k} - F_{N,j}) \langle\langle N, j| W |M, k\rangle = (E_M - E_N) Q_{(N,j),(M,k)} .$$

Due to the way of its derivation, this relation may be understood as an immediate generalization of bi-orthogonality. In particular, we see that within the subspace of a single Sturmian multiplet (i.e., for $M = N$), the left-hand side expression must be a diagonal matrix with respect to its second indices. For our present purposes we shall abbreviate $\langle\langle N, j|W|N, j\rangle \equiv w_{N,j}$.

Non-QES bound-state problems

2.3. Matrix equations. The knowledge of a basis can facilitate the study of many perturbed Hamiltonians via textbook perturbation recipes [F]. The use of the present SES bases might lead to a new progress in this area. Let us recall the respective right and left forms of any Schrödinger non-QES bound state problem with the above-mentioned structure,

$$[H(0) + F W] |\Psi\rangle = E(F) |\Psi\rangle ,$$

$$\langle\langle \Psi| [H(0) + F W] = E(F) \langle\langle \Psi| .$$

Assuming that $F \neq F^{(QES)}$ and using eq. (2.4) we may insert

$$|\Psi\rangle = \sum_{a \in J_{ket}, B \in J_{bra}} |a\rangle R_{a,B} \langle\langle B|\Psi\rangle = \sum_{a \in J_{ket}} |a\rangle h_a ,$$

$$\langle\langle \Psi| = \sum_{a \in J_{ket}, B \in J_{bra}} \langle\langle \Psi|a\rangle R_{a,B} \langle\langle B| = \sum_{B \in J_{bra}} g_B \langle\langle B|$$

and arrive at the double eigen-problem

$$\sum_{b \in J_{ket}, C \in J_{bra}} \langle \langle A | H(F) | b \rangle R_{b,C} \langle \langle C | \Psi \rangle = E \langle \langle A | \Psi \rangle, \quad A \in J_{bra}$$

(plus its – omitted – conjugated companion), i.e., at the two conjugate linear algebraic systems of equations written in terms of the same matrix,

$$(2.8) \quad \mathbf{Z}(E, F) \vec{h} = 0, \quad \vec{g}^\dagger \mathbf{Z}(E, F) = 0$$

$$\mathbf{Z}_{A,b}(E, F) = \langle \langle A | H(0) | b \rangle - E \langle \langle A | b \rangle + F \langle \langle A | W | b \rangle .$$

Now, one could introduce a small parameter $\lambda = F - F^{(QES)}$ and try to construct, say, $E = E(\lambda)$ in the form of a power series in λ .

2.4. Matrix elements. Long before any numerical determination of bound states, we must evaluate *all* the necessary matrix elements as an input. In practice, the latter step is usually the most time-consuming part of the algorithm. One has to optimize it in the present QES setting, therefore.

In the first step we recall eq. (2.6) and eliminate all the matrix elements of $H(0)$. This means that in eq. (2.8) we reduce the costly input to the mere evaluation of the matrix elements of the weakly singular Coulomb potential $W(r) = i/r$,

$$\mathbf{Z}_{A,b}(E, F) = (F - F_A) \langle \langle A | W | b \rangle - (E - E_A) \langle \langle A | b \rangle .$$

In the second step we keep $M \neq N$ (i.e., we stay out of the Sturmian subspaces or diagonal blocks in the matrix \mathbf{Z}) and postulate the absence of a random degeneracy of charges. This means $F_{M,k} \neq F_{N,j}$ so that we are permitted to re-arrange the bi-orthogonality-like relation (2.7) into a further reduction of the necessary input information,

$$\langle \langle N, j | W | M, k \rangle = \frac{E_M - E_N}{F_{M,k} - F_{N,j}} Q_{(N,j),(M,k)} , \quad M \neq N .$$

In this way we arrive at the final form of our linear Schrödinger non-QES algebraic problem for the right eigenvectors,

$$\begin{aligned} w_{N,j} h_{N,j} + \sum_{K(\neq N), p} \frac{E_N - E_K}{F_{N,j} - F_{K,p}} Q_{(N,j),(K,p)} h_{K,p} = \\ = \frac{E - E_N}{F - F_{N,j}} \sum_{M,k} Q_{(N,j),(M,k)} h_{M,k} , \\ j = 1, 2, \dots, L, \quad N = 0, 1, \dots . \end{aligned}$$

For left eigenvectors, the system of equations is very similar though not equivalent,

$$\begin{aligned} w_{N,j} g_{N,j} + \sum_{K(\neq N), p} \frac{E_N - E_K}{F_{N,j} - F_{K,p}} Q_{(K,p),(N,j)} g_{K,p} = \\ = \frac{E - E_N}{F - F_{N,j}} \sum_{M,k} Q_{(M,k),(N,j)} g_{M,k} , \\ j = 1, 2, \dots, L, \quad N = 0, 1, \dots . \end{aligned}$$

To solve any of these systems, say, by a perturbation technique, we just need to know the overlaps Q and the vector of Coulombic elements $w_{N,j}$.

3. CONCLUDING QUESTIONS AND REMARKS

3.1. Does the quantum quasi-exact solvability have a classical analogue? In some studies⁴ a new relationship has been traced between classical and quantum mechanics. Its core may be seen in a correlation between the concepts of *integrability* and *solvability*. Needless to repeat, the former feature plays a key role in classical systems while its versions known as superintegrability and maximal superintegrability acquire more relevance in quantized world.

For many quantum Hamiltonians the latter qualities also seem related to a purely algebraic property of quasi-exact solvability [R]. In such a context, it is extremely exciting to ask questions about the robustness of the latter correlation *and* about the natural ways of the definition of the solvability itself. In a purely pragmatic manner, PT symmetrization could offer here another bridge towards the new solvable models. It is evident that after one weakens the traditional requirement of Hermiticity, many new QES models may be found, indeed [CI, Z6].

In the present paper we have seen that for the D dimensional central Coulomb plus shifted harmonic oscillator its quasi-exact solvability acquires a fairly unusual modified form. A significant difference appears between the quasi-even and quasi-odd states, where only the properties of the latter set remain standard. The facilitated construction and unexpectedly non-numerical character of the former quasi-even family make it similar to the current complete oscillator basis on half-axis. This locates our new “semi-exact” solutions in a gap between their older exact and quasi-exact neighbors.

3.2. Do we need more semi-exactly solvable models? Our presentation of details started from the reduction of the four-term recurrences of our “paper I” [Z4] to their improved three-term form. This was rendered possible by our new ansatz which also proved able to reproduce all the available older results. We underlined that one of the most characteristic shortcomings of the standard QES equations lies in the necessity of solving the adjacent linear algebraic N –dimensional eigenvalue problem of growing size N which, in our particular example, selects the admissible charges.

In this context it is important that we succeeded in revealing the existence of a new, “facilitated” or “semi-exact” elementary solvability emerging when the quasi parity was assumed even. *Infinitely many* of these states acquire the exact polynomial form in each ($= m$ –th) partial wave and at any element $E = E(N)$ of an equidistant set of the energies. In fact, our states form the Sturmian L –plets (numbered by $N = 0, 1, \dots$) at charges $F_{N,k}$ numbered by $k = 1, 2, \dots, L$ where $L = D - 2 + 2m$ is N –independent.

The well known even-parity basis on $L_2(0, \infty)$ (made of Hermite polynomials) is re-obtained in the simplest $L = 1$ special case where the resulting (single) QES charge is zero, $F = 0$. In the first nontrivial $L = 2$ case we have got the two alternative eligible infinite series of states at charges $F = \pm\sqrt{b^2 + 2N}$ (where b is the shift). We have shown that and how the similar sets might serve as a source of the new matrix reformulations and approximation techniques within perturbation theory or variational considerations.

Our main attention has been paid to the generalized QES ansatz admitting solutions with both quasi-parities. Our main result, viz., a new version of the SES

⁴as reviewed, e.g., by Pavel Winternitz in this volume [Wi]

recurrent construction is “almost exact”, first of all, due to its N -independence. In the first few lowest partial waves our infinite families of semi-exact states proved very transparent and close, by their applicability, to the basis of harmonic oscillators. We revealed that their construction is amenable to an efficient approximative treatment even at the higher partial waves via the Taylor-series expansions similar to the common large- N perturbation theory. This might not only initiate the study of the old models in the non-QES domains of couplings but also offer a new motivation for an intensified search for the new SES models.

3.3. Will our new SES bases find efficient practical applications? The matrices of overlaps $Q_{A,b}$ are, presumably, non-diagonal. In the other words, our SES basis states $|N, k\rangle$ and $|N, k'\rangle$ are *not* mutually bi-orthogonal. Any deeper insight in their overlaps would be appreciated in applications, therefore. In the future, this could facilitate, say, a search for a non-Hermitian analogue of the QES + extrapolation trick as suggested by Burrows et al [BC] for the Hermitian sextic anharmonicities.

Our formalism is not yet fully prepared for initiation of any practical numerical or perturbative calculations in the non-QES regime of course. At the same time, we already succeeded in a drastic reduction of the number of the necessary input matrix elements. This supports our belief that our present constructions represent a valuable key step towards deeper understanding of relations between the degree of integrability in classical mechanics and an extent of exact solvability in quantum mechanics. At this moment we have to re-emphasize the definite progress in our understanding of properties of the “forgotten” quasi-even solutions, their quasi-exact solvability of which may be characterized as “significantly facilitated”. They are distributed in the coupling-energy plane along curves which may be interpreted as a common generalization of the usual bound-state straight lines⁵ and the straight lines of the so called Sturmian states⁶. In this sense, their families may be understood as certain generalized Sturmian families defined on some curves in the plane of charge and energy. These sets contain infinitely many elements and definitely resemble and generalize the current types of bases.

3.4. Could the complexification serve as a regularization recipe for SW models? In connection with the strongly singular behaviour of many classically superintegrable models at $r = 0$, a new role of the present weakening of their Hermiticity may be also sought in its parallel regularization effect. In this role, the PT symmetric regularization proved very useful and really vital in the various PT symmetrized versions of the so called supersymmetric (SUSY) quantum mechanics [ZC, B]. For example, in accord with refs. [Z8] the SW Hamiltonians $H = p^2 + r^2 + G/r^2$ may form the *formal* SUSY partners $H_{(L,R)}$. Once their singularity at $r = 0$ is circumvented via the complex shift of the coordinate axis, their spectra remain real and discrete and exhibit still the usual SUSY-isospectrality pattern in the so called unbroken PT symmetry domain with $G > -1/4$.

In [Z9] we have shown that beyond the latter domain, the PT symmetry itself is completely broken but the spectra stay partially real on a SUSY-generated

⁵characterized by the fixed couplings and variable energies and exemplified by the spectrum of the exactly solvable harmonic oscillator or by the Singh’s sextic QES oscillators [SB]

⁶characterized by the variable couplings and fixed energies and exemplified by the exactly solvable zero-order models in ref. [Z7] or by the Flessas’ non-polynomial QES oscillators [WF]

set of complex G . In addition, an alternative SUSY scheme with equal spikes, $G_{(L)} = G_{(R)}$ may be introduced giving an innovated super-Hamiltonian factorized in terms of certain creation and annihilation differential operators of the second order which conserve the quasi-parity and are mutually not adjoint. Together with the original Hamiltonian H the latter two operators commute in accord with the Lie algebra $sl(2, R)$.

All these preliminary results and hints are encouraging and suggest that in the context of the Calogero and/or Smorodinsky-Winternitz superintegrable models with strong barriers the PT symmetry-induced tunneling effects did not say their last word yet.

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4. APPENDICES

A. Charged harmonic oscillator and the concept of quasi-parity

As we already mentioned, the shifted and charged harmonic oscillator of ref. [H] is a characteristic illustrative example of QES model in quantum mechanics. Its Hermitian Schrödinger equation

$$(4.1) \quad \left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + \frac{f}{r} + 2gr + r^2 \right] \varphi(r) = E \varphi(r), \quad r \in (0, \infty)$$

has been replaced by its PT -symmetrized version (1.1) in ref. [Z4]. After the severe restriction of the potential to s -waves in three dimensions (i.e., to $D = 3$ and $\ell = 0$, motivated by technical reasons) we succeeded in confirming its quasi-exact solvability in non-Hermitian regime. In the present continuation of paper [Z4] this result is extended to all the partial waves $\ell = (D-3)/2, (D-3)/2 + 1, \dots$ and dimensions $D = 3, 4, \dots$

The first hint indicating the open possibilities in connection with eq. (4.1) appeared in ref. [Z2] where the concept of quasi-parity has been introduced. One was able to conclude there that all the solutions of the differential Schrödinger equations of the radial type prove separated into quasi-even or quasi-odd states

$\psi_n^{(\pm)}(x)$ in a way which complements the quasi-odd solutions of ref. [Z4] by their present “forgotten” partners with quasi-even symmetry. The key notion of the quasi-parity itself may be defined either via asymptotics or, more easily, via the specific behaviour of the wave functions near the singularity at $r(x) \sim 0$,

$$\psi_n^{(+)}(x) \sim (x - i\varepsilon)^{-\ell}, \quad \psi_n^{(-)}(x) \sim (x - i\varepsilon)^{\ell+1}, \quad x \sim i\varepsilon.$$

The name suggests that the quasi-parity coincides with the ordinary parity after limiting transition $\varepsilon \rightarrow 0$ to a Hermitian Hamiltonian. In this limit, the quasi-even solutions themselves need not lose their normalizability (and may remain physical) whenever the strength of their singular repulsion is sufficiently small (cf. an explicit illustration in ref. [Z8]).

B. QES states with odd quasi-parity

Formula (1.4) for the energies is independent of the charge and was already known to Hautot [H]. Thus, the energy is a constant for all the solutions of eq. (1.3). The multiplets of bound states of such a type⁷ find applications, say, in perturbation theory [Z7]. Up to a free normalization $p_N \neq 0$, their present closed definition

$$(4.2) \quad p_{j-1}^{(-)} = \frac{p_N^{(-)}}{(-A_j)(-A_{j+1}) \dots (-A_N)} \det \begin{pmatrix} B_j & C_j & & & \\ A_{j+1} & B_{j+1} & C_{j+1} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{N-1} & B_{N-1} & C_{N-1} \\ & & & A_N & B_N \end{pmatrix}$$

is unique at any energy E , shift b and charge F .

Our new ansatz (1.2) is sufficiently flexible and reproduces all the older quasi-odd QES solutions of refs. [H] and [Z4]. Indeed, we may mark them by the superscript $(-)$ and characterize their set by the following boundary condition,

$$(4.3) \quad p_0^{(-)} = p_1^{(-)} = \dots = p_{L-1}^{(-)} = 0, \quad p_L^{(-)} \neq 0$$

which reduces the range of the indices in formula (4.2), $j = N, N-1, \dots, L$. We have to guarantee that the QES recurrent recipe terminates, i.e., that the value of $p_{L-1}^{(-)}$ vanishes. This condition has the secular form

$$(4.4) \quad S^{(-)} = \det \begin{pmatrix} B_L & C_L & & & \\ A_{L+1} & B_{L+1} & C_{L+1} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{N-1} & B_{N-1} & C_{N-1} \\ & & & A_N & B_N \end{pmatrix} = 0$$

and its occurrence is characteristic for all QES systems [U]. At every admissible energy it specifies a multiplet of the admissible charges $F = F_{N,j}^{(-)}$ numbered by the second subscript $j = 1, 2, \dots, N-L+1$. At each N and L our secular polynomial is of degree $N-L+1$. Of course, the practical determination of the quasi-odd QES

⁷defined at N different charges $F_{N,j}$ and called Sturmians

eigencharges is a purely numerical procedure for $N > L + 3$. Fortunately, as we have seen in section 1, this shortcoming may be suppressed in the quasi-even case.

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